

GENERALIZED SQUEEZED STATES FOR THE JACOBI GROUP

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ABSTRACT. We analyze the relationship between the covering of the Jacobi group and the squeezed states. We attach some nonclassical states to the Jacobi group. The matrix elements of the Jacobi group are presented.

1. INTRODUCTION

The Jacobi group [1] – the semidirect product of the Heisenberg-Weyl group and the symplectic group – is an important object in the framework of Quantum Mechanics, Geometric Quantization, Optics [2, 3, 4, 5, 6, 7, 8, 9]. The Jacobi group was investigated by physicists under other various names, as “Schrödinger group” [10] or “Weyl-symplectic group” [9]. The squeezed states [2, 3, 4, 5] in Quantum Optics represent a physical realization of the coherent states associated to the Jacobi group. Here we continue the investigation of the Jacobi group started in [11, 12] using Perelomov coherent states [13].

In [11] we have constructed generalized coherent states attached to the Jacobi group, $G_1^J = H_1 \rtimes \mathrm{SU}(1, 1)$, based on the homogeneous Kähler manifold $\mathcal{D}_1^J = H_1/\mathbb{R} \times \mathrm{SU}(1, 1)/\mathrm{U}(1) = \mathbb{C}^1 \times \mathcal{D}_1$. Here \mathcal{D}_1 denotes the unit disk $\mathcal{D}_1 = \{w \in \mathbb{C} \mid |w| < 1\}$, and H_n is the $(2n+1)$ -dimensional real Heisenberg-Weyl (HW) group with Lie algebra \mathfrak{h}_n . In [11] we have also emphasized the connection of our results with those of Berndt and Schmidt [14] and Kähler [15]. In [12] we have considered coherent states attached to the Jacobi group $G_n^J = H_n \rtimes \mathrm{Sp}(2n, \mathbb{R})$, defined on the manifold $\mathcal{D}_n^J = \mathbb{C}^n \times \mathcal{D}_n$, where \mathcal{D}_n is the Siegel ball.

In the present note we underline the connection between the squeezed states in Quantum Optics and the covering of the Jacobi group. In [11] we have considered the action of the Jacobi group on the minimal weight vector $e_0 = \varphi_0 \otimes \phi_{k0}$, where φ_0 is the vacuum vector ($\mathbf{a}\varphi_0 = 0$), while ϕ_{k0} is the minimal weight vector of the positive discrete series representation of the group $\mathrm{SU}(1, 1)$. The standard squeezed states correspond to $e_0 = \varphi_0$ [2]. In the present paper we also give the matrix elements of the Jacobi group acting on $\varphi_n \otimes \phi_{km}$ (*generalized* coherent states attached to the Jacobi group), where φ_n is n -particle (Fock) \mathcal{F} vector, while ϕ_{km} is a normalized vector obtained by the action of $(\mathbf{K}_+)^m$ on ϕ_{k0} , $k \geq 0$, where \mathbf{K}_+ is the rising generator for the group $\mathrm{SU}(1, 1)$. More details about this calculation are given elsewhere [16]. Many particular realizations of the generalized squeezed state for the Jacobi group are known. We recall that the coherent states have been introduced by Schrödinger [17], the squeezed states by Kennard [18] and rediscovered later [2, 3, 4, 5], the displaced squeezed number states by Husimi and Senitzky [19, 20], the squeezed number states by Plebanski [21].

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In the present note we follow the notation and convention of [11]. If π is a representation of a Lie group G with Lie algebra \mathfrak{g} , then we denote $\mathbf{X} = d\pi(X)$, $X \in \mathfrak{g}$.

2. THE JACOBI GROUP AND ITS ALGEBRA

The Jacobi algebra is defined as the the semi-direct sum of the Lie algebra \mathfrak{h}_1 of the Heisenberg-Weyl Lie group and the algebra of the group $SU(1, 1)$, $\mathfrak{g}_1^J = \mathfrak{h}_1 \rtimes \mathfrak{su}(1, 1)$.

The Heisenberg-Weyl ideal $\mathfrak{h}_1 = \langle is1 + xa^\dagger - \bar{x}a \rangle_{s \in \mathbb{R}, x \in \mathbb{C}}$ is determined by the commutation relations

(1)

$$[a, K_+] = a^\dagger; [K_-, a^\dagger] = a; [K_+, a^\dagger] = [K_-, a] = 0; [K_0, a^\dagger] = \frac{1}{2}a^\dagger; [K_0, a] = -\frac{1}{2}a,$$

where a^\dagger (a) are the boson creation (respectively, annihilation) operators, which verify the canonical commutation relations $[a, a^\dagger] = I$, $[a, I] = [a^\dagger, I] = 0$, and $K_{0,+,-}$ are the generators of $SU(1, 1)$ which satisfy the commutation relations:

$$(2) \quad [K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0.$$

We take a representation of G_1^J (cf. [11] and (9)-(11) and Proposition 1 reproduced below) such that the cyclic vector e_0 fulfills simultaneously the conditions

$$(3) \quad \mathbf{a}e_0 = 0, \quad \mathbf{K}_-e_0 = 0, \quad \mathbf{K}_0e_0 = ke_0; \quad k > 0,$$

and we take $e_0 = \varphi_0 \otimes \phi_{k0}$. We consider for $Sp(2, \mathbb{R}) \approx SU(1, 1)$ the unitary irreducible positive discrete series representation D_k^+ with Casimir operator $C = K_0^2 - K_1^2 - K_2^2 = k(k-1)$, where k is the Bargmann index for D_k^+ . The orthonormal canonical basis of the representation space of $SU(1, 1)$ consists of the vectors

$$(4) \quad \phi_{km} = \left[\frac{\Gamma(2k)}{m!\Gamma(2k+m)} \right]^{1/2} (\mathbf{K}_+)^m \phi_{k0}, \quad m \in \mathbb{Z}_+.$$

Also, in the Fock space \mathcal{F} , we have

$$\mathbf{a}\varphi_n = \sqrt{n}\varphi_{n-1}; \quad \mathbf{a}^\dagger\varphi_n = \sqrt{n+1}\varphi_{n+1}, \quad \varphi_n = (n!)^{-\frac{1}{2}}(\mathbf{a}^\dagger)^n \varphi_0; \quad \langle \varphi_{n'}, \varphi_n \rangle = \delta_{nn'}.$$

Perelomov coherent state vectors associated to the group Jacobi G_1^J , based on the manifold \mathcal{D}_1^J , are defined as

$$(5) \quad e_{z,w} = e^{z\mathbf{a}^\dagger + w\mathbf{K}_+} e_0, \quad z, w \in \mathbb{C}, \quad |w| < 1.$$

We introduce the auxiliary operators

$$(6) \quad \mathbf{K}_+ = \frac{1}{2}(\mathbf{a}^\dagger)^2 + \mathbf{K}'_+, \quad \mathbf{K}_- = \frac{1}{2}\mathbf{a}^2 + \mathbf{K}'_-, \quad \mathbf{K}_0 = \frac{1}{2}(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}) + \mathbf{K}'_0,$$

which have the properties

$$(7) \quad \mathbf{K}'_-e_0 = 0, \quad \mathbf{K}'_0e_0 = k'e_0; \quad k = k' + \frac{1}{4},$$

$$(8) \quad [\mathbf{K}'_\sigma, \mathbf{a}] = [\mathbf{K}'_\sigma, \mathbf{a}^\dagger] = 0, \quad \sigma = \pm, 0; \quad [\mathbf{K}'_0, \mathbf{K}'_\pm] = \pm \mathbf{K}'_\pm; \quad [\mathbf{K}'_-, \mathbf{K}'_+] = 2\mathbf{K}'_0.$$

The meaning of the splitting (6) is explained in Theorem 2.6.1 from [14], while the physical consequences of this splitting are briefly discussed in §4. More details are given elsewhere [16]. The positive discrete series corresponds in (7) to $2k' = \text{integer}$.

We introduce the *displacement operator*

$$(9) \quad D(\alpha) = \exp(\alpha \mathbf{a}^\dagger - \bar{\alpha} \mathbf{a}) = \exp(-\frac{1}{2}|\alpha|^2) \exp(\alpha \mathbf{a}^\dagger) \exp(-\bar{\alpha} \mathbf{a}),$$

and the *unitary squeezed operator* of the D_+^k representation of the group $SU(1, 1)$, $\underline{S}(z) = S(w)$ ($w = \frac{z}{|z|} \tanh(|z|)$, $\eta = \ln(1 - |w|^2)$):

$$(10a) \quad \underline{S}(z) = \exp(z \mathbf{K}_+ - \bar{z} \mathbf{K}_-), \quad z \in \mathbb{C};$$

$$(10b) \quad S(w) = \exp(w \mathbf{K}_+) \exp(\eta \mathbf{K}_0) \exp(-\bar{w} \mathbf{K}_-), \quad |w| < 1.$$

We introduce also [11] the *generalized squeezed coherent state vector*

$$(11) \quad \Psi_{\alpha, w} = D(\alpha) S(w) e_0, \quad e_0 = \varphi_0 \otimes \phi_{k0}.$$

We recall some properties of the coherent states associated to the group G_1^J , proved in [11]:

Proposition 1. *The generalized squeezed coherent state vector (11) and Perelomov coherent state vector (5) are related by the relation*

$$(12) \quad \Psi_{\alpha, w} = (1 - w\bar{w})^k \exp(-\frac{\bar{\alpha}}{2}z) e_{z, w}, \text{ where } z = \alpha - w\bar{\alpha}.$$

Perelomov coherent state vector (5) was calculated in [11] as

$$(13) \quad e_{z, w} = E(z, w) \varphi_0 \otimes e^{w\mathbf{K}'_+} \phi_{k0},$$

$$(14) \quad E(z, w) \varphi_0 = e^{z\mathbf{a}^\dagger + \frac{w}{2}(\mathbf{a}^\dagger)^2} \varphi_0 = \sum_{n=0}^{\infty} \frac{P_n(z, w)}{(n!)^{1/2}} \varphi_n,$$

$$(15) \quad P_n(z, w) = n! \sum_{p=0}^{[\frac{n}{2}]} \left(\frac{w}{2}\right)^p \frac{z^{n-2p}}{p!(n-2p)!}.$$

The base of functions $f_{nks}(\alpha, w) = \langle e_{\bar{\alpha}, \bar{w}}, \varphi_n \otimes \phi_{ks} \rangle$, where $k = k' + 1/4$, $2k' = \text{integer}$, $n, s = 0, 1, \dots$, in which the Bergman kernel (22) can be expanded, is:

$$(16) \quad f_{nks}(\alpha, w) = f_{k's}(w) \frac{P_n(\alpha, w)}{\sqrt{n!}}, \quad f_{ls}(w) = \sqrt{\frac{\Gamma(s+2l)}{s!\Gamma(2l)}} w^s, \quad |w| < 1.$$

The composition law in the Jacobi group $G_1^J = HW \rtimes SU(1, 1)$ is

$$(17) \quad (g_1, \alpha_1, t_1) \circ (g_2, \alpha_2, t_2) = (g_1 \circ g_2, g_2^{-1} \cdot \alpha_1 + \alpha_2, t_1 + t_2 + \text{Im}(g_2^{-1} \cdot \alpha_1 \bar{\alpha}_2)),$$

where $g^{-1} \cdot \alpha = \bar{a}\alpha - b\bar{\alpha}$, and $g \in SU(1, 1)$ is parametrized as

$$(18) \quad g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 - |b|^2 = 1,$$

Let $h = (g, \alpha) \in G_1^J$, $\pi(h)_k = T(g)_k D(\alpha)$, $g \in SU(1, 1)$, $\alpha \in \mathbb{C}$, and let $x = (z, w) \in \mathcal{D}_1^J = \mathbb{C} \times \mathcal{D}_1$. Then we have the formula:

$$(19) \quad \pi(h)_k \cdot e_{z, w} = (\bar{a} + \bar{b}w)^{-2k} \exp(-\lambda_1) e_{z_1, w_1},$$

$$(20) \quad \lambda_1 = \frac{\bar{b}(z + z_0)^2}{2(\bar{a} + \bar{b}w)} + \bar{\alpha}(z + \frac{z_0}{2}), \quad z_0 = \alpha - \bar{\alpha}w,$$

$$(21) \quad z_1 = \frac{\alpha - \bar{\alpha}w + z}{\bar{b}w + \bar{a}}; \quad w_1 = \frac{aw + b}{\bar{b}w + \bar{a}}.$$

The space of functions \mathcal{H}_K attached to the reproducing kernel

$$(22) \quad K = (1 - w\bar{w})^{-2k} \exp \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2w}{2(1 - w\bar{w})},$$

consists of square integrable functions with respect to the scalar product

$$(23) \quad (f, g)_k = \frac{4k - 3}{2\pi^2} \int_{\alpha \in \mathbb{C}; |w| < 1} \frac{\bar{f}(\alpha, w)g(\alpha, w)}{(1 - |w|^2)^3 K(w, \bar{w})} d^2\alpha d^2w.$$

3. THE COVERING OF $SU(1, 1)$

If in (19) we take $\alpha = 0$, then $T(g)_k e_w = (\bar{a} + \bar{b}w)^{-2k} e_{w_1}$, which corresponds to the positive discrete series representation of $SU(1, 1)$ [22]

$$(24) \quad [T(g)_k]f(z) = (\bar{a} + \bar{b}z)^{-2k} f\left(\frac{az + b}{\bar{b}z + \bar{a}}\right), \quad g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1, 1),$$

for the Hilbert space $\mathcal{H}_{2,k}(\mathcal{D}_1)$, $2k = \text{integer}$, of holomorphic functions on \mathcal{D}_1 with respect to the scalar product [22]

$$(25) \quad (f, g)_k = \frac{2k - 1}{\pi} \int_{|w| < 1} \bar{f}(w)g(w)(1 - |w|^2)^{2k-2} d^2w.$$

If in (19) we put $w = 0$, we get $D(\alpha)e_z = e^{-\frac{|\alpha|^2}{2} - z\bar{\alpha}}e_{z+\alpha}$, which corresponds to the Segal-Bargmann-Fock representation of the Heisenberg group (see e.g. (1.71) in [23]).

(19) corresponds to the continuous unitary representation (π_k, \mathcal{H}_K) of G_1^J

$$(26) \quad (\pi(h)_k \cdot f)(x) = (\bar{a} + \bar{b}w)^{-2k} \exp(-\lambda_1) f(x_1), \quad f \in \mathcal{H}_K,$$

identified in [11] with the result established in [1, 14].

The functions f_{nl} in (16) form an orthonormal base in $\mathcal{H}_{2,k}(\mathcal{D}_1)$, $k > 1/2$, and if $f = \sum_{n=0}^{\infty} a_n z^n$, $g = \sum_{n=0}^{\infty} b_n z^n$, $|z| < 1$, then

$$(27) \quad (f, g)_k = \sum_{n=0}^{\infty} \frac{\Gamma(2k)\Gamma(n+1)}{\Gamma(2k+n)} \bar{a}_n b_n.$$

The representations of the group $\widetilde{SU(1, 1)}$ have been considered in [24, 25, 26, 27]. For the universal covering group of $SU(1, 1)$, we use a class of Hilbert spaces indexed by a parameter $k \in \mathbb{R}$, $0 < |k| < 1/2$ (cf. Sally [25, 26]).

Let now $\mathcal{H}_{2,k}(\mathcal{D}_1)$ be the Hilbert space of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ holomorphic in \mathcal{D}_1 with norm

$$\|f\|_k^2 = \sum_{n=0}^{\infty} \frac{\Gamma(2k)\Gamma(n+1)}{\Gamma(2k+n)} |a_n|^2 < \infty.$$

According to Bargmann ((4.3) in [22]), starting from the matrix $g \in \widetilde{\text{SU}(1, 1)}$ parameterized by (18), an element \tilde{g} of the covering group $\widetilde{\text{SU}(1, 1)}$ is parametrized by $(\gamma, w) \in \mathbb{R} \times \mathcal{D}_1$ via the relations

$$(28) \quad a = e^{i\omega}(1 - |\gamma|^2)^{-1/2}, \quad b = e^{i\omega}\gamma(1 - |\gamma|^2)^{-1/2}, \quad |\gamma| < 1,$$

and we have $\omega \in (-\pi/2, \pi/2]$, $\omega \in (-\pi, \pi]$, $\omega \in \mathbb{R}$, for, respectively, $\text{SO}^\dagger(1, 2)$, $\text{Sp}(1, \mathbb{R}) \approx \text{SU}(1, 1)$, $\text{SU}(1, 1)$.

Remark 1. *The representation (29) of $\widetilde{\text{SU}(1, 1)}$ is a continuous, irreducible unitary representation with respect with the scalar product (27)*

$$(29) \quad [T(\tilde{g})_k f](z) = e^{2ik\omega}(1 - |\gamma|^2)^k(1 + \bar{\gamma}z)^{-2k}f\left(\frac{az + b}{\bar{b}z + \bar{a}}\right), \quad k \in \mathbb{R}, \quad 0 < k < 1/2.$$

Proof. Lemma (1.3.1) and Theorems (2.5.2), (2.5.3) in the reference [25] are used. Here we verify just that $(T(g)_k f_1, f_2)_k = (f_1, T(g^{-1})_k f_2)_k$, for $g \in \text{SU}(1, 1)$ and $f_{1,2}$ from the orthonormal set (16). We take $f_1 = a_{kn}z^n$, $f_2 = a_{kN}z^N$, where $a_{kn}^2 = \frac{\Gamma(n+2k)}{n!\Gamma(2k)}$. Then $T(g)_k f_1 = A_{kn}(1 - x)^{-q}(1 + y)^n$, where $A_{kn} = a_{kn}(\bar{a})^{-2k-n}b^n$, $x = -\lambda z$, $y = \frac{z}{\lambda}$, $\lambda = \frac{b}{a}$, $q = 2k + n$, and $g \in \text{SU}(1, 1)$ is the matrix (18). Then

$$(T(g)_k f_1, f_2)_k = \bar{A}_{kn}a_{kN}\left(\sum_{m=0}^{\infty} b_{qm}x^m \sum_{p=0}^n C_p^n y^p, z^N\right)_k,$$

where $b_{qm} = \frac{\Gamma(q+m)}{m!\Gamma(q)}$ and $C_k^n = \frac{n!}{k!(n-k)!}$. With the orthogonality of the system (16), we get

$$(T(g)_k f_1, f_2)_k = \frac{\bar{A}_{kn}}{a_{kN}} \sum_{m=0}^{\infty} (-\lambda)^m b_{qm} \sum_{p=0}^n C_p^n \bar{\lambda}^{-p} \delta_{m+p, N}.$$

Similarly, we have ($Q = 2k + N$)

$$(f_1, T(g^{-1})_k f_2)_k = \frac{a_{kN}}{a_{kn}} a^{-Q}(-b)^N \sum_{M=0}^{\infty} b_{QM} \left(\frac{\bar{b}}{a}\right)^M \sum_{p=0}^N C_p^N \left(-\frac{\bar{a}}{b}\right)^p \delta_{n, p+M}.$$

So, we have (we take $N - n \geq 0$)

$$(T(g)_k f_1, f_2)_k = \frac{a_{kn}}{a_{kN}} a^{-q} \bar{a}^N \bar{b}^{n-N} \sum_{m=N-n}^N b_{qm} C_{N-m}^n (-|b/a|)^{2m},$$

$$(f_1, T(g^{-1})_k f_2)_k = \frac{a_{kN}}{a_{kn}} a^{-Q}(-b)^N \left(-\frac{\bar{a}}{b}\right)^n \sum_{M=0}^n b_{QM} C_{n-M}^N (-|b/a|)^{2M}.$$

With the change of variables $m - (N - n) = M$, it is easy to check that the last two expressions are identical.

Remark 2. *It can be checked out that the differential operators $\mathbb{K}_- = \frac{\partial}{\partial z}$, $\mathbb{K}_0 = k + z\frac{\partial}{\partial z}$, $\mathbb{K}_+ = 2kz + z^2\frac{\partial}{\partial z}$ corresponding to the generators of $\text{SU}(1, 1)$ have the adequate hermitian conjugate properties with respect to the scalar product $(\cdot, \cdot)_k$ (27), $k > 0$.*

4. SQUEEZED STATES AND JACOBI GROUP

The standard squeezed states [2, 3, 4, 5] correspond to the action of Jacobi group on the extremal weight vector $e_0 = \varphi_0$. This corresponds to take zero the action of \mathbf{K}' in the splitting (6). This is the group which admits the so called *Schrödinger-Weil* representation π_{SW}^m with character $\psi^m(x) = e^{2\pi i mx}$, $m \in \mathbb{R}$, considered in the mentioned Theorem 2.6.1 in [14], where x is in the center of the Heisenberg-Weyl algebra. The part of the representation corresponding to the covering group \tilde{G}_1^J is called “Weil representation” at p. 23 in [14].

We see that we have $\mathbf{K}_0 \varphi_{2p} = (p + \frac{1}{4}) \varphi_{2p}$, $\mathbf{K}_0 \varphi_{2p+1} = (p + \frac{3}{4}) \varphi_{2p+1}$, and irreducible representations with $k = \frac{1}{4}$, $k = \frac{3}{4}$ of the covering group $\widetilde{SU}(1, 1)$ must be considered. The vacuum squeezed state contains only even Fock states.

So, dealing with squeezed states, we have to consider the covering of the Jacobi group G_1^J , $\tilde{G}_1^J = HW \rtimes \widetilde{SU}(1, 1)$.

The orthonormal basis of \mathcal{H}_K for G_1^J in the realization $\mathbf{K}_+ = \frac{1}{2}(\mathbf{a}^\dagger)^2$ in the splitting (6), where the reproducing kernel K is given by (22) with $k = 1/4$, consists of polynomials $(n!)^{-1/2} P_n$, $n = 0, 1, \dots$. Instead of the formula (19) for G_1^J ($2k' = \text{integer}$), we get for \tilde{G}_1^J ($k' > 0$) the formula

$$(30) \quad \pi(\tilde{h})_k \cdot e_{z,w} = e^{2ik\omega} (1 - |\gamma|^2)^k (1 + \bar{\gamma}w)^{-2k} \exp(-\lambda_1) e_{z_1, w_1}.$$

5. MATRIX ELEMENTS FOR THE JACOBI GROUP

Now we briefly present the main steps in the calculation of the matrix elements of G_1^J with respect to the considered representation:

$$(31) \quad \langle \varphi_{n'} \otimes \phi_{km'} | D(\alpha) S(w) | \varphi_n \otimes \phi_{km} \rangle.$$

More details are given elsewhere [16].

Firstly, we re-obtain the matrix elements of the Heisenberg-Weyl group [28, 29]:

$$(32) \quad \langle \varphi_m | D(\alpha) | \varphi_n \rangle = \sqrt{\frac{n!}{m!}} \alpha^{m-n} L_n^{m-n}(|\alpha|^2) \exp(-|\alpha|^2/2), \quad m \geq n,$$

where m, n are non-negative integers and L_n^s are the associated Laguerre polynomials.

Next we calculate the matrix elements $S(w)_{km'm} = \langle \phi_{km'} | S(w) | \phi_{km} \rangle$. We use the relations:

$$(33a) \quad \mathbf{K}_0 \phi_{km} = (k + m) \phi_{km},$$

$$(33b) \quad \mathbf{K}_+ \phi_{km} = [(m + 1)(m + 2k)]^{1/2} \phi_{km+1},$$

$$(33c) \quad \mathbf{K}_- \phi_{km} = [m(m + 2k - 1)]^{1/2} \phi_{km-1}, \quad m > 0, m \in \mathbb{Z}_+.$$

We introduce (33c) in the expression of $S(w)$ and we get

$$(34) \quad \begin{aligned} \exp(-\bar{w}\mathbf{K}_-) \phi_{km} &= \sum_{p=0}^m \frac{(-\bar{w})^p}{p!} \mathbf{K}_-^p \phi_{km} \\ &= \sum_{p=0}^m \left[\frac{m!}{(m-p)!} \frac{\Gamma(2k+m)}{\Gamma(2k+m-p)} \right]^{1/2} \frac{(-\bar{w})^p}{p!} \phi_{km-p}. \end{aligned}$$

Then we apply successively (33a) and (33b) and we have ($m' \geq m$):

$$(35) \quad \begin{aligned} S(w)_{km'm} &= \left[\frac{m'! \Gamma(2k+m')}{m! \Gamma(2k+m)} \right]^{1/2} \frac{w^{m'-m}}{(m'-m)!} (1-w\bar{w})^{k+m} \\ &\times F\left(-m, 1-2k-m; m'-m+1; \frac{-w\bar{w}}{1-w\bar{w}}\right). \end{aligned}$$

The matrix elements of the group $SU(1, 1)$ for the discrete series representations have been calculated in [22], and the case corresponding to the $\widetilde{SU(1, 1)}$ for $k = 1/4, 3/4$ in [30].

Using Kummer's formula $F(a, b; c; \frac{z}{z-1}) = (1-z)^a F(a, c-b; c; z)$, we put (35) in the form

$$(36) \quad S(w)_{km'm} = \left[\frac{m'! \Gamma(2k+m')}{m! \Gamma(2k+m)} \right]^{\frac{1}{2}} \frac{w^{m'-m} (1-w\bar{w})^k}{(m'-m)!} F(-m, 2k+m'; m'-m+1; |w|^2)$$

for $m' \geq m$, while for $m \geq m'$, we have $S_{km'm}(w) = S_{kmm'}(-\bar{w})$.

Now we calculate the matrix elements (31). Starting from the splitting (6), we introduce the auxiliary operator

$$(37) \quad \begin{aligned} S(w, w') &= \exp\left[\frac{w}{2}(a^\dagger)^2\right] \exp\left[\frac{\eta}{2}(a^\dagger a + \frac{1}{2})\right] \exp\left(-\frac{\bar{w}}{2}a^2\right) \\ &\otimes \exp(w'K'_+) \exp(\eta'K'_0) \exp(-\bar{w}'K'_-), \\ \eta &= \ln(1-w\bar{w}), \quad \eta' = \ln(1-w'\bar{w}'). \end{aligned}$$

The matrix elements $S_{km'm}(w)$ are given by (35), (36). (35) is a particular case of formula (10.28) in [22]. Moreover, the formula (10.28) in [22] can be re-obtained from our (35) using the relation

$$(38) \quad T(g)_k \phi_{km} = \left(\frac{a}{|a|} \right)^{2(k+m)} S\left(\frac{b}{\bar{a}}\right) \phi_{km}, \quad g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1, 1).$$

Using the identification $\varphi_{2n+\epsilon} \leftrightarrow \phi_{1/4+\epsilon/2;n}$, ($\epsilon = 0$ or 1), the matrix elements (31) of the Jacobi group G_1^J are obtained taking $w = w'$ in formula (39):

$$(39) \quad \begin{aligned} \langle \varphi_{n'} \otimes \phi_{km'} | D(\alpha) S(w, w') | \varphi_{2s+\epsilon} \otimes \phi_{km} \rangle &= S_{km'm}(w') \\ &\times \sum_{s' \geq 0} \langle \varphi_{n'} | D(\alpha) | \varphi_{2s'+\epsilon} \rangle S_{1/4+\epsilon/2, s's}(w). \end{aligned}$$

The matrix elements $\pi(h)$ are obtained from (39) taking into account (38).

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